# Algorithmic mathematics in a technical university: different ways to comprehend the material

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# Current problem of mathematical training of engineers

For a long time, the methodology of teaching mathematics in technical universities was not a problem for students.

The teacher presented the subject in accordance with the logical structure of the subject, and the students understood the lectures and comprehended the basic ideas of the subject, more or less regardless of how “clear” or “incomprehensible” the subject was presented by the teacher.

The current level of school preparation has led to the fact that **most students have lost such a quality as “the ability to comprehend the material**.”

The situation is reminiscent of the problem 100 years ago (on a different level), which Academician Alexei Nikolaevich **Krylov wrote about in his memoirs**.

# Styles of presenting mathematics to engineers: deductive, inductive or...?

Subconsciously we contrast two styles of presenting material.
Simplified, they can be characterized as follows:

**Deductive:**

definition ➔ theorem ➔ corollary ➔ example

**Inductive:**

example ➔ generalization ➔ math calculations and reasoning ➔ definition

Let's consider an example of the presentation of linear algebra from Strang's textbook “Linear Algebra and Its Applications” and try to determine the style of the author’s approach.

# Methodological features of the mathematics course for engineers

| An example from the plan for algebra and analytical geometry in the 1st year: | An example of their Strang on the same material (Strang G. Linear algebra and its applications. Mir. Moscow. 1980.) |
|---|---|
| 1. Calculation of determinants, matrix rank (4 hours)<br>2. Matrix multiplication. Inverse Matrix (4 hours)<br>3. Study of systems of linear equations, Cramer's theorem. Gauss method for solving systems of linear equations (4 hours) | 1. Example of using the Gaussian elimination method<br>2. Matrix notation and matrix multiplication<br>3. Equivalence of the Gaussian elimination method and triangular matrix decomposition |

# Case Study

Different methodological concepts are presented in different columns: in the first case we go from determinants to linear equations, in the second - in the reverse order (and determinants do not appear at all yet).

**In both cases, it is possible to implement both a deductive style of presentation and an inductive one**. For example, the introduction of a determinant can begin with a general definition, or it can be done with an analysis of 2x2 linear systems, isolating expressions that are called 2x2 determinants, studying their properties and transferring them to determinants of higher orders.

However, **this is not Strang's idea**: just as in the inductive approach, he starts with linear equations, but does not move away from this representation of the ideas of linear algebra into determinants. Why?

# Investment approach

Technically oriented students tend to build on the concepts and skills they have. We can classify them as common sense.

Solving equations is a separate line in the school mathematics course. There are equations in one form or another in both physics and chemistry.

In his presentation, Strang gradually identifies entities that are natural for a person who has already solved equations: instead of operating with individual numbers, operating with columns is introduced. The matrix is considered first as a union of columns. Thus, **at each step of presentation, new concepts are tied to the already existing ideas of the listeners**. For example, a linear space of columns and the concept of rank as the dimension of this space arise naturally.

**Marvin Minsky calls this approach of learning an investment approach.**

# An experiment to analyze the influence of existing ideas on the formation of new concepts

**Description of the experiment.** As part of the topic “Binary Relations,” students were introduced to the concept of transitive closure. Two algorithms for constructing a transitive closure were analyzed: through the multiplication of adjacency matrices and the Warshall algorithm.

After this, one of the groups was given the task: **within an hour and a half, to propose a definition of transitive reduction** (a formal definition was not given, only an idea - as in a sense the inverse operation of transitive closure.

They had to set themselves various tasks and solve them. There was a hint that they can consider different types of graphs, conduct experiments, formulate hypotheses, create algorithms and justify their correctness, count the number of certain graphs, and estimate the complexity of algorithms.

# Key facts related to the task

The expected results of the study should have approximately corresponded to the results presented in the article A. V. Aho, M. R. Garey and J. D. Ullman. The Transitive Reduction of a Directed Graph /  Siam J. Comput. Vol. 1, No. 2, June 1972:

**Abstract**. We consider economical representations for the path information in a directed graph. A directed graph G’ is said to be a transitive reduction of the directed graph G provided that **(i) G has a directed path from vertex u to vertex v if and only if G has a directed path from vertex u to vertex v, and (ii) there is no graph with fewer arcs than G’ satisfying condition (i).** Though **directed graphs with cycles may have more than one such representation**, we select a **natural canonical representative** as the transitive reduction for such graphs. It is shown that the time complexity of the best algorithm for finding the transitive reduction of a graph is the **same as the time to compute the transitive closure** of a graph or to **perform Boolean matrix multiplication**.

# Experiment results (without details)

Four different non-equivalent definitions have been presented:

1. **The first is “classic”, possibly taken from the Internet**
2. **The second is due to the absence of triples of the form**: aRb & bRc & aRc
3. **The third is the uniqueness of the path from one vertex to another.**
4. **The fourth is a generalization of transitive reduction,** the introduction of minimum and maximum reduction (in fact, related to the number of edges k in the path from vertex u to vertex v, the presence of which justifies the reduction of edge (u;v))

The students were also divided according to the type of graphs being considered: some considered directed graphs, others undirected graphs.

Some students working with directed graphs understood the main thing in solving the problem - the **presence or absence of cycles**, but formally associated this condition with antisymmetry.

# Detailing results-1

Based on the above reasoning, we can conclude what ideas the students relied on.

**Undirected graphs and spanning trees**. The task was set to find transitive reductions and their number for a complete undirected graph. In this case, the author did not consider the edges of the graph as bidirectional arcs, but used the concept of reachability. **The author justified that a spanning tree would be a transitive reduction and calculated their number for the complete graph**. In this case, in fact, the student was dealing with cycles, but his model was simplified and **the result turned out to be completely consistent with the above-mentioned Aho-Garey-Ullman theorem on the canonical representation of a transitive reduction in the presence of a cycle.**

**Here we can recall the words of Poincare: “Kepler would not have invented his laws if Tycho Brahe’s instruments had been more accurate.”**

# Detailing results-2

The most frequently cited algorithm for constructing a transitive reduction, called naive by one of the students, is an **erroneous algorithm** (from ITMO Wikipedia [https://neerc.ifmo.ru/](https://neerc.ifmo.ru/) ), **which only works for transitive acyclic relations, although neither constraint is specified** (antisymmetric is written instead of acyclic):

```
R':=R;
FOR x∈V
    FOR  y∈V
        FOR z∈V
            If xRy and yRz and xRz, then remove the (x;z) from R'
```

**Obviously, Warshall's algorithm served as a template for reasoning, that is, the students tried to reverse the transitive closure algorithm.**

# Detailing results-3

One of the students used reachability considerations in conjunction with the idea of Warshall's algorithm and realized that it was necessary **to remove not only those edges that could be duplicated by a two-edge path, but also by k-edge paths.**

He used the property of the k-th powers of the adjacency matrix, proven in the lecture - reachability along paths of k edges (an algorithm has an error).

“A - nxn $B=A^2$

FOR i∈{1,...,n}

  FOR j∈{i,...,n}

    If A[i][j]=1 $\wedge$ B[i][j]=1

      A[i][j]:=0; $B:=A^2$; A[j][i]:=0”

For some reason, the author calls this reduction maximum, and then formulates the **correct idea for constructing a transitive reduction algorithm** (without caring about the efficiency of the algorithm): “For the minimum we will use the same algorithm, only for **B we will take it equal to $A^i$**, i∈[2;k]...”

# Detailing results-4

Beautiful solutions to auxiliary problems were given:

- “In a complete graph of N vertices, the edges are randomly oriented. It turned out that there are no cycles in the graph. Find a transitive reduction of this graph” (**it was shown that the condition of the absence of cycles in a complete graph implies transitivity and therefore the relation is a relation of linear order**)

- “It was hypothesized that by arranging directions in an undirected complete graph, it is possible to ensure that no edges are removed during transitive reduction. **During the analysis, the hypothesis was rejected, an algorithm for constructing a counterexample was obtained (it is impossible to arrange directions for n≥4 vertices in this way.”**

# Model for the formation of the concept of transitive reduction

We will consider a group of students as a whole. Let us highlight the most common ideas from which the new concept was built:

- **The concept of transitive closure**
- **Definition of transitivity**
- **Beliefs about reachability (relation)**
- **The idea of transitive closure (Warshall's algorithm)**
- **Ideas about the “identical” orientation of the edges** (“from left to right”). This idea was the basis for the requirement of acyclicity, which some students noted explicitly, although they did not consider graphs with cycles, some (erroneously) associated with the antisymmetry, others called strict order.

# How does the investment principle work?

**Students try patterns that match their external attributes and try to somehow reconstruct them:** add something or try to reverse it. At the same time, they test themselves using examples or simple judgments that they have already "appropriated."

In parallel, **restrictions are added in accordance with the "bad" examples**, or (if there are several tasks) a narrower class of objects is selected for which the proposed patterns does not contradict those "bad" examples. These restrictions may appear implicitly, that is, not recognized by the decision maker.

At the next stage, **attempts are made to refute the hypotheses. Counterexamples appear for which the theory needs to be "refined."**

# Sets on which the presented reasoning is correct (different for different students)

**Transitive acyclic relations**

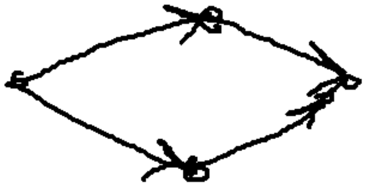
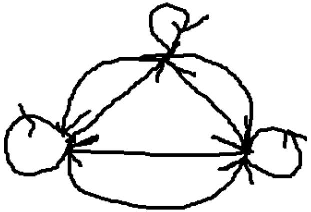
In fact, all students are repelled by the presence of transitivity. Apparently, having in mind the transitive closure algorithm and trying to reverse it, they start with the result of this algorithm.

**Undirected complete graphs**

Apparently, ideas about reachability are at work here. The definition of a transitive reduction as a graph that is reachable along a single path assumes representations that are true for undirected graphs.

**Arbitrary graphs**

# “Unstated” counterexamples

| | |
|---|---|
|  The algorithm does not work on non transitive relations, although the graph is acyclic |  Transitive contraction is not equivalent to path uniqueness |
|  The algorithm does not work on transitive relations for a graph with cycles |  Removing arcs during reduction results in an error |

# Conclusions

1. The introduced concept is comprehended if and when the **student has the opportunity (has time) to build a model by which he can predict the approximate formulation** of the introduced concept and connect all the elements of the definition with existing ideas.
2. Verification of a hypothetical definition is carried out by performing “trial” mental operations on some example (or a similar concept), which is selected as characteristic of the subject area associated with the new concept. However, **the pattern used is usually not equivalent to the concept**.
3. After these stages, through self-criticism and external **criticism through counterexamples**, ideas are corrected by adding new patterns and restrictions.

Taken together, all this constitutes a meaningful concept.
Psychologists have shown that the third stage does not have an effect if it precedes the previous two stages.